Telescope conjectures and Bousfield lattices for localized categories of spectra

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This talk is based on the material in arXiv: 1307.3351.

Tensor triangulated category theory

Let T be a well generated triangulated category with a closed symmetric monoidal product $\wedge$ and unit $\mathbb{1}$. Assume T has all coproducts. Assume $\mathbb{1}$ is strongly dualizable, a.k.a. rigid.

(X is *strongly dualizable* if $F(X, \mathbb{1}) \wedge Y \xrightarrow{\sim} F(X, Y)$ for all Y.)

Examples:

- the derived category $D(R)$ of a commutative ring R
- the (p-local) stable homotopy category $\mathcal{S}$
- the stable module category $\mathsf{StMod}(kG)$ of a finite group G

Types of questions we can ask about T:

1. Classify thick subcategories of compact objects.
   (*thick*: triangulated and closed under taking retracts)
2. Classify localizing subcategories.
   (*localizing*: triangulated and closed under taking coproducts)
3. Compute Bousfield lattice.
   $$\left(\begin{array}{l} \langle X \rangle = \{W \mid W \wedge X = 0\} \\ \langle X \rangle \leq \langle Y \rangle \text{ if } W \wedge Y = 0 \Rightarrow W \wedge X = 0 \end{array} \right)$$
4. Classify smashing localization functors.
   ($L : \mathsf{T} \to \mathsf{T}$ localization that commutes with coproducts)

This talk is about localized categories of spectra.

Let Z be a spectrum in $\mathcal{S}$. Let $L_Z : \mathcal{S} \to \mathcal{S}$ be Bousfield localization with acyclics $\langle Z \rangle$, i.e. L_Z-acyclics are Z_*-acyclics.

Let $\mathcal{L}_Z$ be the essential image of L_Z; these are the L_Z-local spectra. $\mathcal{L}_Z$ has the structure of a tensor triangulated category. The triangles are the same as in $\mathcal{S}$, but

- $X \wedge_{\mathcal{L}_Z} Y = L_Z(X \wedge Y)$
- $X \coprod_{\mathcal{L}_Z} Y = L_Z(W \coprod Y)$
- $L_Z\mathbb{1}$ is the unit of $\mathcal{L}_Z$
- $\mathcal{L}_Z = \mathsf{loc}(L_Z\mathbb{1})$.

Note: $\mathcal{L}_Z$ is equivalent to the Verdier quotient $\mathcal{S}/\langle Z \rangle$.

Examples: harmonic category, $K(n)$-local, $E(n)$-local, BP-local, $H\mathbb{F}_p$-local, or IS^0-local categories. (IS^0 the Brown- Comenetz dual of the sphere.)

For example, Hovey and Strickland looked at $K(n)$-local and $E(n)$-local categories from this perspective.

In the $K(n)$-local category, there are exactly two localizing subcategories, and hence two smashing localization functors. The Bousfield lattice is $\{\langle 0\rangle, \langle K(n)\rangle\}$.

In the $E(n)$-local category, there are $n+1$ smashing localization functors. Every localizing subcategory is a Bousfield class, and the Bousfield lattice is isomorphic to the lattice of subsets of $\{0, 1, 2, ..., n\}$.

Telescope conjectures in $\mathcal{S}$:

Every finite spectrum $F(n+1)$ yields a localization functor L_n^f, called finite localization away from $F(n+1)$, with acyclics

$$\langle T(0) \vee \cdots T(n) \rangle = \mathsf{loc}(F(n+1)).$$

These are smashing localizations.

Also, $L_{E(n)} = L_n : \mathcal{S} \to \mathcal{S}$, with $\langle E(n) \rangle = \langle K(0) \vee \cdots \vee K(n) \rangle$ as acyclics is smashing.

Telescope conjecture.

$$L_{n-1}^f X \cong L_n X \text{ for all } X \text{ and } n \geq 0.$$

Well, sort of.

Are these all the smashing localization functors on $\mathcal{S}$?

Every set of compact objects yields a smashing localization, and this holds in any tensor triangulated category.

Generalized smashing conjecture (GSC).

Every smashing localization comes from localization away from a set of compact objects.

This has been asked and answered (and often called the telescope conjecture) in many different categories, e.g. $D(R)$ and $\mathsf{StMod}(kG)$. If true in $\mathcal{S}$, this would imply the telescope conjecture.

Theorem.

Let T be a well generated tensor triangulated category such that $\mathbb{1}$ is strongly dualizable and $\mathsf{loc}(\mathbb{1}) = \mathsf{T}$. Let $A = \{B_\alpha\}$ be a set of strongly dualizable objects. Then there exists a smashing localization functor $L : \mathsf{T} \to \mathsf{T}$ with $\mathrm{Ker}\, L = \mathsf{loc}(A)$.

Strongly dualizable generalized smashing conjecture (SDGSC).

Every smashing localization comes from localization away from a set of strongly dualizable objects.

Note: in $\mathcal{S}$ (or any T with $\mathbb{1}$ compact), compact $\iff$ strongly dualizable.

Example: harmonic category. Let $\mathcal{H} = \mathcal{L}_Z$, with $Z = \bigvee_{i \geq 0} K(i)$. The harmonic category has no nonzero compact objects.

Proposition.

For every $n \geq 0$, there exists a smashing localization functor $l_n : \mathcal{H} \to \mathcal{H}$. Every nontrivial smashing localization on $\mathcal{H}$ is l_n for some n.

Proposition.

For every $n \geq 0$, l_n is localization away from $L_Z F(n)$, which is strongly dualizable.

Corollary.

In $\mathcal{H}$, the GSC fails but the SDGSC holds.

Example: the $H\mathbb{F}_p$-local category.

Theorem.

In the $H\mathbb{F}_p$-local category, there are exactly two smashing localizations. The GSC fails but the SDGSC holds. The Bousfield lattice is $\{\langle 0\rangle, \langle H\mathbb{F}_p\rangle\}$.

Example: the IS^0-local category.
(IS^0 is the Brown-Comenetz dual of the sphere.)

Theorem.

In the IS^0-local category, there are exactly two smashing localizations. The GSC fails but the SDGSC holds. The Bousfield lattice is $\{\langle 0\rangle, \langle L_{IS^0}S^0\rangle\}$.

Breaking news:

In what categories is every localizing subcategory a Bousfield class? This was conjectured by Hovey and Palmieri for $\mathcal{S}$. This was proved for $D(R)$ when R is Noetherian, and recently for $\mathsf{StMod}(kG)$.

Let $\mathsf{T} = \mathcal{L}_{H\mathbb{F}_p}$ be the $H\mathbb{F}_p$-local category. Both $H\mathbb{F}_p$ and the Moore spectrum $M(p)$ are $H\mathbb{F}_p$-local. Furthermore, $[H\mathbb{F}_p, M(p)]_* = 0$ and $[M(p), M(p)]_* \neq 0$.

The Bousfield lattice has two elements, $\langle 0 \rangle = \mathsf{T}$ and $\langle \mathbb{1} \rangle = \{0\}$.

(Thanks to Mark Hovey for pointing out on MathOverflow that...) The collection of $H\mathbb{F}_p$-local X such that $[X, M(p)]_* = 0$ is a localizing subcategory of T that is not a Bousfield class.

The end. Thanks for listening. See arXiv: 1307.3351 for more info.