Equivariant Algebraic K-theory

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Algebraic K-theory functor

$$\begin{tikzcd} Rings \arrow[r, bend left=5, "K_i"] & Groups \end{tikzcd}$$

Algebraic K-theory functor

$$
\begin{tikzcd}
\mathit{Rings} \arrow[rr, bend left, "K_i"] \arrow[dr, "K"'] & & \mathit{Groups} \\
& \mathit{Spaces}/\mathit{Spectra} \arrow[ur, "\pi_i"'] &
\end{tikzcd}
$$

The input category *Rings* can be replaced with *ExactCat* or *WaldhausenCat*.

Why do we care about algebraic K-theory?

Algebraic K-theory is the meeting ground for various other subjects such as

- algebraic geometry,
- number theory,
- algebraic topology,

and it encodes deep information about these.

Why do we care about algebraic K-theory?

Example (Number theory)

Let $\mathbb{Q}(\zeta_p)^+$ be the maximal real subfield of $\mathbb{Q}(\zeta_p)$ and $h_{\mathbb{Q}(\zeta_p)^+}$ its class number. Then

$$p \nmid h_{\mathbb{Q}(\zeta_p)^+} \iff K_{4i}(\mathbb{Z}) = 0 \text{ for all} i.$$

Example (Manifold theory)

The Waldhausen K-theory spectrum of a manifold M decomposes as

$$A(M) \simeq \Sigma^\infty M_+ \times WH^{PL}(M),$$

where the second factor encodes information about pseudo-isotopies on M.

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G-actions in algebraic K-theory, a motivating example

Suppose E/F is a (finite) Galois extension with Galois group G. Then G acts on KE and

$$(KE)^G = KF.$$

We have a map from the fixed points to the homotopy fixed points

$$(KE)^G \to (KE)^{hG}$$

induced by

$$(KE)^G = Map(*, KE)^G \to Map(EG, KE)^G.$$

Initial Quillen-Lichtenbaum conjecture

Conjecture (Lichtenbaum-Quillen)

The map

$$KF \to KE^{hG}$$

is an equivalence after p-completion.

Theorem (Thomason)

The conjecture becomes true only after inverting a "Bott element" and reducing mod a prime power.

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My goal

The goal is to tell an equivariant story...

Question: What kind of equivariant spectra are there?

- **naive** G-spectra (spectra with G-action)
- **genuine** G-spectra (indexed on G-representations, can suspend and desuspend with respect to representation spheres S^V, give rise to $RO(G)$-graded cohomology theories)

Naive G-spectra are the ones that arise naturally when G acts on the input.
But, genuine G-spectra are the "right" objects for equivariant stable homotopy theory.

My goal

GOAL: Encode a "naive" action on the input ring (or category) as a genuine G-spectrum.

$$\begin{tikzcd}
G\text{-Rings} \arrow[rr] \arrow[dr, "\mathbb{K}_G"] & & \text{Mackey functors} \\
& \text{Genuine } G\text{-Spectra} \arrow[ur, "\underline{\pi}_i"] &
\end{tikzcd}$$

Equivariant algebraic K-theory functor

Theorem (M.)

There is a functor $\mathbb{K}_G : G$-rings $\to$ genuine G-spectra, which applied to

- *the topological ring $\mathbb{C}$ with trivial G-action $\rightsquigarrow$ yields KU_G*
- *the topological ring $\mathbb{R}$ with trivial G-action $\rightsquigarrow$ yields KO_G*
- *the topological ring $\mathbb{C}$ with $\mathbb{Z}/2$ conjugation action $\rightsquigarrow$ yields Atiyah's KR*

Theorem, continued

We recover the original Quillen-Lichtenbaum conjecture

Theorem (M.)

Let E/F be a Galois extension with group G. The map of spectra

$$\mathbb{K}_G(E)^G \to \mathbb{K}_G(E)^{hG}$$

from fixed points to homotopy fixed points of the genuine G-spectrum $\mathbb{K}_G(E)$ is equivalent to the map

$$K(F) \to K(E)^{hG},$$

where $K(E)^{hG}$ denotes the homotopy fixed points of the naive spectrum $K(E)$.

In particular, $\mathbb{K}_G(E)^G \simeq K(\mathbb{Q})$ for any Galois extension $E/\mathbb{Q}$!

Construction of the functor $\mathbb{K}_G$

2 steps:

1. good space level definition
2. equivariant delooping of this space

Step 1

Recall (or take as definition):

$$BGL(R)^+ \simeq \text{group completion of } \coprod_n \underbrace{BGL_n(R)}_{\substack{\text{classifying spaces of} \\ \text{principal } GL_n(R)\text{-bundles}}} .$$

Idea: Replace by equivariant $(G, GL(R) \rtimes G)$-bundles when G acts on R.

Equivariant bundle theory

Let $\mathcal{C}$ be a G-category. Define:
$\tilde{G} :=$ translation category of G
$\mathcal{C}at(\tilde{G}, \mathcal{C}) :=$ functor category, with G acting by conjugation

NOTES: $\mathcal{C}at(\tilde{G}, \mathcal{C}) \simeq \mathcal{C}$ nonequivariantly, and $B\tilde{G} \simeq EG$.

Theorem (Guillou, May, M.)

Suppose G acts on Π. The canonical map

$$B\mathcal{C}at(\tilde{G}, \tilde{\Pi}) \to B\mathcal{C}at(\tilde{G}, \Pi),$$

is a universal principal $(G, \Pi \rtimes G)$-bundle.

Definition of K-theory space, equivariant plus-construction

Suppose R is a G-ring.

Definition

Define $K_G(R)$ to be the equivariant group completion of $B\mathcal{C}at(\tilde{G}, \coprod GL_n(R))$.

Question: Is this an equivariant infinite loop space?

Step 2: Equivariant delooping

$\mathcal{C}at(\tilde{G}, \coprod GL_n(R))$ turns out to be an algebra over the genuine E_∞-operad as defined by Guillou-May.

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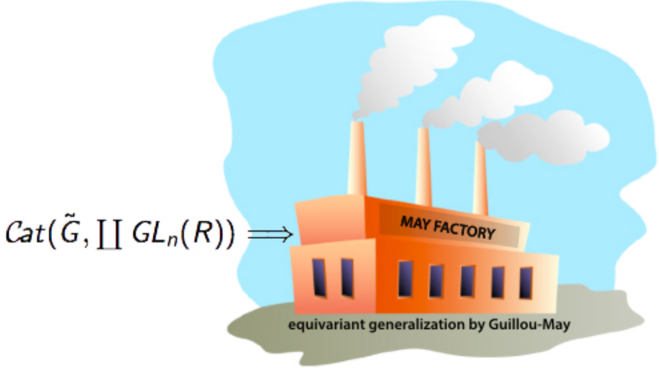

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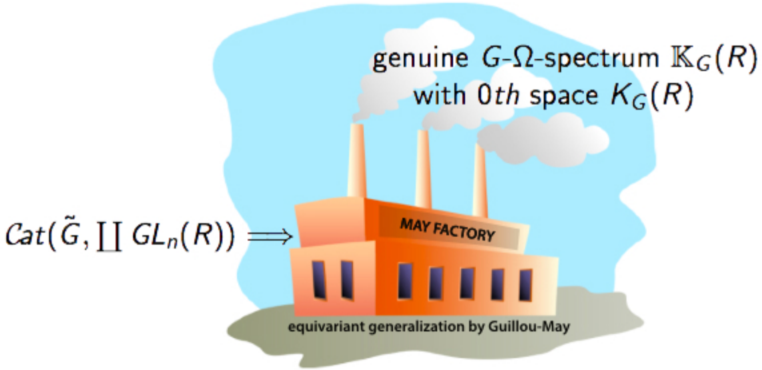
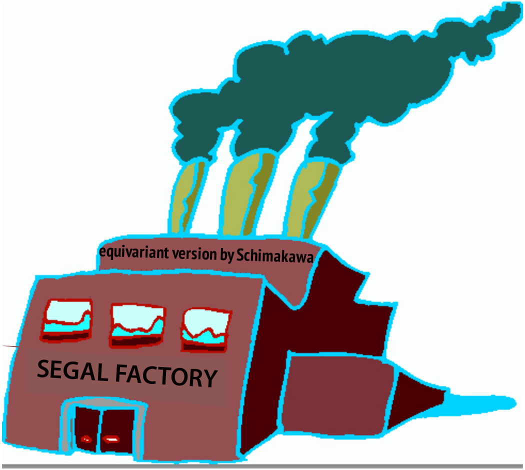


Equivariant Segal machine

Issue: There is another equivariant infinite loop space machine, the equivariant version of Segal's machine, developed by Schimakawa.



Comparison theorem

Question: Do these equivariant infinite loop space machines produce equivalent outputs?

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Theorem (May-M.-Osorno)

Exact/ Waldhausen G-categories

For an exact/Waldhausen G-category $\mathcal{C}$, $\mathcal{C}at(\tilde{G}, \mathcal{C})$ is also exact/Waldhausen.

Definition

Define $K_G(\mathcal{C})$ to be $\Omega QB\mathcal{C}at(\tilde{G}, \mathcal{C})$ if $\mathcal{C}$ is an exact G-category and $\Omega|wS_\bullet\mathcal{C}at(\tilde{G}, \mathcal{C})|$ if $\mathcal{C}$ is a Waldhausen G-category.

Theorem (M.)

$+ = Q = S_\bullet$.

Conjecture

$K_G(\mathcal{C})$ is an infinite loop G-space.

Future Directions

- K-theory of G-Waldhausen categories; do we have a splitting

$$A_G(M) \simeq \Sigma_G^\infty M_+ \times Wh_G^{PL}(M),$$

  where $Wh_G^{PL}(M)$ encodes information about equivariant pseudo-isotopies of a G-manifold M?
- Carlsson's program for describing the K-theory of a field in terms of the representation theory of the absolute Galois group. (His map occurs as the fixed point map of the constructions I gave above.)

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Thank you!!!