MU^* and BP^*

$$MU^*_{(p)} = \mathbb{Z}_{(p)}\,[U_1, U_2, U_3, \ldots]$$

$$MU^{-*} \otimes \mathbb{Q} \cong H\mathbb{Q}_*(MU) \cong \mathbb{Q}[m_1, m_2, m_3, \ldots]$$

$$[\mathbb{C}P^n] \in MU^{-2n}$$

Under the Hurewicz map to rational homology

$$[\mathbb{C}P^n] \mapsto (n+1)m_n.$$

$$\mathbb{Q}\left[[\mathbb{C}P^1], [\mathbb{C}P^2], [\mathbb{C}P^3], \ldots\right] \xhookrightarrow{\cong} MU^* \otimes \mathbb{Q}$$

MU^* and BP^*

$$MU^* \xrightarrow{r_*} BP^*$$

$$MU^* \otimes \mathbb{Q} \longrightarrow BP^* \otimes \mathbb{Q}$$

$$m_i \mapsto \begin{cases} 0 & \text{if } i \neq p^k - 1 \\ \ell_k & \text{if } i = p^k - 1 \end{cases}$$

$$\mathbb{Q}[\ell_1, \ell_2, \ell_3, \ldots] \xhookrightarrow{\cong} BP^* \otimes \mathbb{Q}$$

$$r_*[\mathbb{C}P^{p^k-1}] = p^k \ell_k \in BP^{-2(p^k-1)} \otimes \mathbb{Q}$$

MU^* and BP^*

$$BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

Hazewinkel generators:

$$\ell_1 = \frac{v_1}{p}, \quad \ell_2 = \frac{v_2}{p} + \frac{v_1^{1+p}}{p^2},$$

$$\ell_3 = \frac{v_3}{p} + \frac{v_1 v_2^p + v_2 v_1^{p^2}}{p^2} + \frac{v_1^{1+p+p^2}}{p^3}, \text{ etc.}$$

Araki generators:

$$\ell_1 = \frac{v_1}{p - p^p}, \text{ etc.}$$

MU^* and BP^*

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!]$$
$$MU^*(BC_p \times \mathbb{C}P^\infty) \cong MU^*[\![x, \xi]\!]/[p]\xi$$

$$BP^*(\mathbb{C}P^\infty) \cong BP^*[\![x]\!]$$
$$BP^*(BC_p \times \mathbb{C}P^\infty) \cong BP^*[\![x, \xi]\!]/[p]\xi$$

$\log_{BP}$, $\exp_{BP}$, and formal sum

$$\log_{BP}(t) = t + \ell_1 t^p + \ell_2 t^{p^2} + \cdots$$
$$\exp_{BP}(t) = \log_{BP}^{-1}(t)$$

$$\xi +_{BP} x = \exp_{BP}\left(\log_{BP}(\xi) + \log_{BP}(x) \right) = \xi + x + \cdots$$
$$[i]\xi = i\xi + \cdots$$

P_{C_p}

$$r_* P_{C_p}(x) = \prod_{i=0}^{p-1} \big([i]\xi +_{BP} x\big) = \sum_{i \geq 0} a_i(\xi) x^{i+1}$$

This defines the a_i, and these give an expression for MC_n:

$$MC_n = a_0^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k]$$

$$a_0(\xi) = \xi$$

$$a_1(\xi) = 1 - v_1\xi + v_1^2\xi^2 + \left(-2v_1^3 - 2v_2\right)\xi^3 + \left(3v_1^4 + 4v_1 v_2\right)\xi^4 + \left(-4v_1^5 - 6v_1^2 v_2\right)$$

$$a_2(\xi) = v_1^2\xi + \left(-4v_1^3 - 3v_2\right)\xi^2 + \left(10v_1^4 + 11v_1 v_2\right)\xi^3 + \left(-21v_1^5 - 28v_1^2 v_2\right)\xi^4 + ($$

$$a_3(\xi) = \left(-2v_1^3 - 2v_2\right)\xi + \left(10v_1^4 + 11v_1 v_2\right)\xi^2 + \left(-34v_1^5 - 43v_1^2 v_2\right)\xi^3 + \left(101v_1^6 +\right.$$

$$a_4(\xi) = \left(3v_1^4 + 4v_1 v_2\right)\xi + \left(-21v_1^5 - 28v_1^2 v_2\right)\xi^2 + \left(101v_1^6 + 164v_1^3 v_2 + 34v_2^2\right)\xi^3$$

$$a_5(\xi) = \left(-4v_1^5 - 6v_1^2 v_2\right)\xi + \left(43v_1^6 + 75v_1^3 v_2 + 18v_2^2\right)\xi^2 + \left(-275v_1^7 - 551v_1^4 v_2 -\right.$$

$$a_6(\xi) = \left(6v_1^6 + 12v_1^3 v_2 + 4v_2^2\right)\xi + \left(-88v_1^7 - 190v_1^4 v_2 - 89v_1 v_2^2 - 14v_3\right)\xi^2 + (68$$

$$a_7(\xi) = \left(-10v_1^7 - 24v_1^4 v_2 - 14v_1 v_2^2 - 4v_3\right)\xi + \left(169v_1^8 + 420v_1^5 v_2 + 257v_1^2 v_2^2 + 4\right.$$

$$a_8(\xi) = \left(15v_1^8 + 40v_1^5 v_2 + 28v_1^2 v_2^2 + 8v_1 v_3\right)\xi + \left(-312v_1^9 - 880v_1^6 v_2 - 688v_1^3 v_2^2 -\right.$$

$a_0(\xi) = 2\xi^2 - 2v_1\xi^4 + 8v_1^2\xi^6 - 40v_1^3\xi^8 + \left(170v_1^4 - 170v_2\right)\xi^{10}$

$a_1(\xi) = 3\xi - 8v_1\xi^3 + 36v_1^2\xi^5 - 216v_1^3\xi^7 + \left(1148v_1^4 - 944v_2\right)\xi^9$

$a_2(\xi) = 1 - 9v_1\xi^2 + 63v_1^2\xi^4 - 491v_1^3\xi^6 + \left(3336v_1^4 - 2331v_2\right)\xi^8 + \left(-19299v_1^5 +\right.$

$a_3(\xi) = -3v_1\xi + 53v_1^2\xi^3 - 606v_1^3\xi^5 + \left(5466v_1^4 - 3396v_2\right)\xi^7 + \left(-40124v_1^5 + 892\right.$

$a_4(\xi) = 21v_1^2\xi^2 - 435v_1^3\xi^4 + \left(5547v_1^4 - 3248v_2\right)\xi^6 + \left(-53343v_1^5 + 109971v_1v_2\right.$

$a_5(\xi) = 3v_1^2\xi - 179v_1^3\xi^3 + \left(3588v_1^4 - 2142v_2\right)\xi^5 + \left(-47382v_1^5 + 94662v_1v_2\right)\xi^7$

$a_6(\xi) = -38v_1^3\xi^2 + \left(1454v_1^4 - 994v_2\right)\xi^4 + \left(-28406v_1^5 + 58352v_1v_2\right)\xi^6 + \left(3223\right.$

$a_7(\xi) = -3v_1^3\xi + \left(341v_1^4 - 324v_2\right)\xi^3 + \left(-11256v_1^5 + 25956v_1v_2\right)\xi^5 + \left(179916v\right.$

$a_8(\xi) = \left(36v_1^4 - 72v_2\right)\xi^2 + \left(-2748v_1^5 + 8268v_1v_2\right)\xi^4 + \left(67120v_1^6 - 518984v_1^2v\right.$

$$a_0(\xi) = 24\xi^4 - 1680v_1\xi^8 + 370008v_1^2\xi^{12} - 123486336v_1^3\xi^{16} + 49940181504v_1^4\xi$$
$$a_1(\xi) = 50\xi^3 - 5430v_1\xi^7 + 1551072v_1^2\xi^{11} - 636927168v_1^3\xi^{15} + 306533455680v$$
$$a_2(\xi) = 35\xi^2 - 7328v_1\xi^6 + 2893808v_1^2\xi^{10} - 1508394320v_1^3\xi^{14} + 880153410800$$
$$a_3(\xi) = 10\xi - 5498v_1\xi^5 + 3207450v_1^2\xi^9 - 2188580410v_1^3\xi^{13} + 1576841873306$$
$$a_4(\xi) = 1 - 2550v_1\xi^4 + 2370055v_1^2\xi^8 - 2186482212v_1^3\xi^{12} + 1981785971805v_1^4$$
$$a_5(\xi) = -750v_1\xi^3 + 1237150v_1^2\xi^7 - 1600089600v_1^3\xi^{11} + 1861052456328v_1^4\xi^{15}$$
$$a_6(\xi) = -130v_1\xi^2 + 469174v_1^2\xi^6 - 889462830v_1^3\xi^{10} + 1357095174226v_1^4\xi^{14} -$$
$$a_7(\xi) = -10v_1\xi + 129998v_1^2\xi^5 - 383662650v_1^3\xi^9 + 787791379990v_1^4\xi^{13} - 1310$$
$$a_8(\xi) = 25850v_1^2\xi^4 - 129787730v_1^3\xi^8 + 369983450960v_1^4\xi^{12} - 78629987651049$$

$\langle p \rangle \xi$

$$\langle 2 \rangle = 2 - \xi v_1 + 2\xi^2 v_1^2 + \xi^3 \left(-8v_1^3 - 7v_2\right) + \xi^4 \left(26v_1^4 + 30v_1 v_2\right) + \xi^5 \left(-84v_1^5 - 11\right.$$

$$\langle 3 \rangle = 3 - 8\xi^2 v_1 + 72\xi^4 v_1^2 - 840\xi^6 v_1^3 + \xi^8 \left(9000v_1^4 - 6560v_2\right) + \xi^{10} \left(-88992v_1^5 +\right.$$

$$\langle 5 \rangle = 5 - 624\xi^4 v_1 + 390000\xi^8 v_1^2 - 341094000\xi^{12} v_1^3 + 347012281200\xi^{16} v_1^4 - 38$$

$$a_0(\xi) \equiv \xi$$
$$a_1(\xi) \equiv 1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + \left(v_1^6 + v_1^3 v_2 + v_2^2\right)\xi^6 + v_1^4 v_2 \xi^7 + \left(v_1^8 + v_1^2 v_2^2 + v_1\right.$$
$$a_2(\xi) \equiv v_1^2\xi + v_2\xi^2 + v_1 v_2\xi^3 + v_1^5\xi^4 + v_1^3 v_2\xi^5 + \left(v_1^4 v_2 + v_1 v_2^2\right)\xi^6 + \left(v_1^8 + v_1^2 v_2^2 +\right.$$
$$a_3(\xi) \equiv v_1^4\xi^2 + \left(v_1^6 + v_1^3 v_2 + v_2^2\right)\xi^4 + v_1^4 v_2\xi^5 + \left(v_1^8 + v_1^5 v_2\right)\xi^6 + \left(v_1^9 + v_1^6 v_2 + v_2^3\right)$$
$$a_4(\xi) \equiv v_1^4\xi + \left(v_1^6 + v_1^3 v_2\right)\xi^3 + \left(v_1^4 v_2 + v_1 v_2^2 + v_3\right)\xi^4 + \left(v_1^5 v_2 + v_1 v_3\right)\xi^5 + \left(v_1^9 +\right.$$
$$a_5(\xi) \equiv v_1^6\xi^2 + \left(v_1^7 + v_1^4 v_2 + v_1 v_2^2\right)\xi^3 + \left(v_1^8 + v_1^2 v_2^2 + v_1 v_3\right)\xi^4 + \left(v_2^3 + v_1^2 v_3\right)\xi^5 +$$
$$a_6(\xi) \equiv \left(v_1^7 + v_1 v_2^2\right)\xi^2 + \left(v_1^8 + v_1^2 v_2^2 + v_1 v_3\right)\xi^3 + v_2^3\xi^4 + \left(v_1^{10} + v_2 v_3\right)\xi^5 + \left(v_1^8 v_2\right.$$
$$a_7(\xi) \equiv v_1^9\xi^3 + \left(v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2\right)\xi^5 + \left(v_1^3 v_2^3 + v_1^5 v_3\right)\xi^6 + \left(v_1^{13} + v_1^{10} v_2 + v_1^4 v_2^3\right.$$
$$a_8(\xi) \equiv v_1^8\xi + v_1^9\xi^2 + \left(v_1^{10} + v_1^4 v_2^2\right)\xi^3 + \left(v_1^{11} + v_1^2 v_2^3 + v_1^4 v_3\right)\xi^4 + \left(v_1^6 v_2^2 + v_1^2 v_2 v_3\right.$$

$$a_0(\xi) \equiv 2\xi^2 + v_1\xi^4 + 2v_1^3\xi^8 + \left(v_1^4 + v_2\right)\xi^{10} + 2v_1^5\xi^{12} + v_1^2 v_2\xi^{14} + \left(v_1^7 + v_1^3 v_2\right)\xi^{16}$$

$$a_1(\xi) \equiv 0$$

$$a_2(\xi) \equiv 1 \quad + v_1^7\xi^{14} + v_1^4 v_2\xi^{16} + \left(v_1^5 v_2 + v_1 v_2^2\right)\xi^{18} + 2v_1^6 v_2\xi^{20} + (2v_1^7 v_2 + 2v_1^3 v_2$$

$$a_3(\xi) \equiv 0$$

$$a_4(\xi) \equiv 2v_1^3\xi^4 + \left(v_1^4 + v_2\right)\xi^6 + 2v_1^5\xi^8 + v_1^2 v_2\xi^{10} + \left(v_1^7 + v_1^3 v_2\right)\xi^{12} + (2v_1^9 + 2v_1^5 v$$

$$a_5(\xi) \equiv 0$$

$$a_6(\xi) \equiv v_1^3\xi^2 + 2v_2\xi^4 + \left(v_1^5 + v_1 v_2\right)\xi^6 + \left(v_1^6 + 2v_1^2 v_2\right)\xi^8 + \left(2v_1^7 + v_1^3 v_2\right)\xi^{10} + (v$$

$$a_7(\xi) \equiv 0$$

$$a_8(\xi) \equiv v_1^9\xi^{12} + 2v_1^{10}\xi^{14} + \left(2v_1^{11} + v_1^7 v_2\right)\xi^{16} + \left(v_1^{12} + 2v_1^8 v_2 + 2v_1^4 v_2^2 + 2v_2^3\right)\xi^{18}$$

$P_{C_p} \bmod \langle p \rangle \qquad p = 5$

$$
\begin{aligned}
a_0(\xi) &\equiv 4\xi^4 + v_1\xi^8 + 4v_1^5\xi^{24} + \left(v_1^6 + v_2\right)\xi^{28} \pmod{(\xi)^{33}} \\
a_1(\xi) &\equiv 0 \\
a_2(\xi) &\equiv 0 \\
a_3(\xi) &\equiv 0 \\
a_4(\xi) &\equiv 1 \pmod{(\xi)^{33}} \\
a_5(\xi) &\equiv 0 \\
a_6(\xi) &\equiv 0 \\
a_7(\xi) &\equiv 0 \\
a_8(\xi) &\equiv 4v_1^5\xi^{16} + \left(v_1^6 + v_2\right)\xi^{20} + 4v_1^9\xi^{32} \pmod{(\xi)^{33}}
\end{aligned}
$$

$$MC_1(\xi) \equiv \xi^2 v_1^2 + \xi^3 v_2 + \xi^4 \left(v_1^4 + v_1 v_2\right) + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^8 + v_1 v_3\right) + \xi^9 \left(v_1^9 + v_1^6 v_2\right.$$

$$MC_2(\xi) \equiv \xi^6 \left(v_1^6 + v_2^2\right) + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^5 v_2 + v_1 v_3\right) + \xi^9 v_2^3 + \xi^{10} \left(v_1^4 v_2^2 + v_1 v_2^3\right) +$$

$$MC_3(\xi) \equiv \xi^6 v_1^6 + \xi^7 \left(v_1^4 v_2 + v_1 v_2^2\right) + \xi^8 \left(v_1^8 + v_1^5 v_2 + v_1 v_3\right) + \xi^{10} \left(v_1^{10} + v_1^7 v_2 + v_1^4 v_2^2\right.$$

$$MC_4(\xi) \equiv \xi^{10} v_1^4 v_2^2 + \xi^{11} \left(v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2 + v_1^4 v_3\right) + \xi^{12} \left(v_1^9 v_2 + v_1^3 v_2^3 + v_2^4\right) + \xi^{13}$$

$MC_2(\xi) \equiv v_1^3\xi^8 + 2v_2\xi^{10} + \left(v_1^5 + v_2 v_1\right)\xi^{12} + 2v_1^2 v_2\xi^{14} + 2v_1^7\xi^{16} + \left(2v_1^8 + v_2^2\right)\xi^{18} + \left(v_2 v\right.$

$MC_4(\xi) \equiv 2v_1^9\xi^{22} + 2v_1^{10}\xi^{24} \pmod{\xi^{26}}$

The obstructions MC_n $\quad p = 5$

$$MC_4(\xi) \equiv \xi^{32} v_1^5 + 4\xi^{36} v_2 \pmod{\xi^{40}}$$

$$MC_8(\xi) \equiv 9a_0^6 \left[2v_1^{16}\xi^{64} + (3v_1^{17} + 4v_1^{11} v_2)\xi^{68}\right] \pmod{\xi^{72}}$$

Conclusion

At the primes 2, 3, and 5, Quillen's map is not H_∞.

Theorem (J.-N.)

At the primes 2, 3, and 5, there are no p-universal orientations of H_∞ ring spectra from MU to BP.

Conjecture

$MC_{2(p-1)}$ is non-zero mod $\langle p \rangle$, and the first non-zero coefficient is ...??
$a_{(p-1)} \equiv 1$ mod ...??

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At the primes 2, 3, and 5, Quillen's map is not H_∞.

Theorem (J.-N.)

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Conjecture

$MC_{2(p-1)}$ is non-zero mod $\langle p \rangle$, and the first non-zero coefficient is ...??
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Thank You!