INVERTIBILITY IN BICATEGORIES

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Abstract. We aim to develop a calculational foothold on Picard and Brauer groups in generalized contexts by investigating invertibility. Aside from introduction of basic definitions and foundations, the essential point is the characterization of generalized Azumaya objects. We give tilting theory as an application of this characterization.

Note: this is a preprint draft

1. Introduction

There is considerable interest throughout topology and algebraic-geometry in developing generalizations of classical algebra. The Picard group and the Brauer group are two starting points for a large body of algebraic work, and in this paper we consider how this theory generalizes to modern contexts. To differentiate between results which hold formally and those which are special to the classical algebraic setting, we adopt a general, bicategorical perspective. First, this allows us to discern the formal relationships among concepts from the rich classical setting. Additionally, it allows us to clarify some of the algebra by reframing intricate algebraic structures in elementary (although no less deep) categorical terms. Furthermore, the categorical setting guides our translation of these concepts into homotopical or derived settings.

Let A be a 0-cell of an autonomous monoidal bicategory $\mathscr{D}$ with unit R. Let A^e denote the 0-cell $A \otimes A^{op}$, and let A_r be the 1-cell $A^e \twoheadrightarrow R$ induced by the unit 1-cell for A. Our results are summarized below.

Theorem. *The following statements are equivalent:*

- *i.* A_r *is an invertible 1-cell.*
- *ii.* *a)* A_r *is right-dualizable and*
  - *b) the evaluation* $\mathrm{sHom}(A_r, A^e) \otimes_R A \to A^e$ *is an isomorphism and*
  - *c) the action of* R *induces* $R \cong \mathrm{sHom}(A_r, A_r)$.
- *iii.* *a)* A_r *is left-dualizable and*
  - *b) the evaluation* $A \otimes_{A^e} \mathrm{tHom}(A, R) \to R$ *is an isomorphism and*
  - *c) the action of* A^e *induces* $A^e \cong \mathrm{tHom}(A_r, A_r)$.
- *iv.* A_r *is left-dualizable and there exists a 0-cell* B *such that* $A \otimes_R B$ *is Morita equivalent to* R.

If $\mathscr{D}$ *is a triangulated bicategory, then these are equivalent to the following:*

- *v.* *a)* A_r *is right-dualizable and*
  - *b) the action of* R *induces* $R \cong \mathrm{sHom}(A_r, A_r)$ *and*
  - *c)* A^e *is left-*A_r*-local.*
- *vi.* *a)* A_r *is left-dualizable and*
  - *b) the action of* A^e *induces* $A^e \cong \mathrm{tHom}(A_r, A_r)$ *and*
  - *c)* R *is right-*A_r*-local.*

Section 2 introduces the categorical notation we will use, and the additional bicategorical structures with which we work. Morita theory is a fundamental starting point for developing both the Picard group and the Brauer group; in Subsection 2.4 we explain how Morita theory can be reframed in elementary terms by the Yoneda Lemma. Section 3 develops the connection between Morita, Picard, and Brauer theory by reviewing basic definitions about duality and invertibility. In Subsection 3.2 we prove a characterization result (3.8) generalizing the equivalent conditions which define Azumaya algebras. This is the result that initiates calculational accessibility of the classical theory, and the same is true in more general contexts. We give a topologically inspired extension (4.14) of this result in Section 4, after introducing and developing the

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necessary triangulated structures in our context. As corollaries, we have the results of Rickard and Schwede-Shipley, giving sufficient conditions for certain Morita equivalences in derived and topological settings.

In future work, we hope to continue developing calculational approaches to Picard and Brauer groups in this general setting.

2. BICATEGORIES

We assume the reader is familiar with basic definitions for bicategories, however we do not assume the reader is an expert. We begin by reviewing some important examples and relevant additional structures. A precise and concise introduction can be found in [Lei98], while [Lac07] provides a more expanded guide. A thorough introduction developed for similar purposes can be found in [Joh08, §3 and §5]

We use arrows such as $f : M \to M'$ to denote that f is a 2-cell with source M and target M', and slashed arrows such as $M : A \nrightarrow B$ to denote that M is a 1-cell with source A and target B. We use $\circ$ or juxtaposition to denote vertical composition of 2-cells, and $\odot$ to denote horizontal composition of 1-cells and of 2-cells.

2.1. **Examples.** Let R be a commutative ring. The collection of R-algebras is the collection of 0-cells for the bicategory $\mathscr{M}_R$. The 1-cells of $\mathscr{M}_R(A, B)$ are the B-A-bimodules, and the 2-cells are bimodule homomorphisms. The horizontal composition, $\odot$, is given by the tensor product: If $M \in \mathscr{M}_R(B, C)$ and $N \in \mathscr{M}_R(A, B)$, then

$$M \odot N = M \otimes_B N \in \mathscr{M}_R(A, C).$$

If, more generally, R is a commutative DG algebra, we have the bicategory DG_R, defined similarly: the 0-cells are DG R-algebras, 1-cells are DG bimodules, and 2-cells are maps of bimodules. We also have the derived bicategory $\mathscr{D}_R$, with the same 0-cells as DG_R, but the category of 1- and 2-cells between two 0-cells is the derived category of bimodules.

Still more generally, if R denotes a commutative ring spectrum we likewise have a bicategory of R-algebra spectra, bimodule spectra, and bimodule morphisms. This bicategory is denoted $\mathscr{S}_R$. Like DG_R, $\mathscr{S}_R$ has a derived bicategory, and we extend the notation $\mathscr{D}_R$ to include the case that R is a spectrum also. Note that if R is a commutative DG algebra and HR its Eilenberg-Mac Lane spectrum, then we have $\mathscr{D}_R(A, B) \simeq \mathscr{D}_{HR}(HA, HB)$ for all DG algebras A and B.

Any cocomplete symmetric monoidal category with unit R gives rise to a bicategory whose 0-cells are monoids and 1-cells are bimodules. Colimits are required to construct the $\odot$, just as they are in the cases of tensor and smash above.

2.2. **Monoidal bicategories.** A monoidal bicategory can be defined as a tricategory with one object. In practical terms, this means that the bicategory is equipped with an additional monoidal product on 0-, 1, and 2-cells, satisfying reasonable associativity and unit constraints. In $\mathscr{M}_R$, the monoidal product is $\otimes_R$; in $\mathscr{S}_R$, it is $\wedge_R$. In these examples, the monoidal product is symmetric, and hence these are symmetric monoidal bicategories. More generally, if $\mathscr{C}_R$ is a cocomplete monoidal category with unit R, and $\mathscr{B}_R$ the bicategory of monoids and bimodules in $\mathscr{C}_R$, then $\mathscr{B}_R$ is a symmetric monoidal bicategory with monoidal product induced by that of $\mathscr{C}_R$.

2.3. **Closed structure.** A *closed structure* for a bicategory, $\mathscr{B}$, defines right adjoints for $\odot$. For a 1-cell M, the right adjoint to $- \odot M$ is called "right-hom", or "source-hom", and denoted $\mathrm{sHom}(M, -)$. The adjoint to $M \odot -$ is called "left-hom", or "target-hom", and denoted $\mathrm{tHom}(M, -)$. The adjunctions are written as

$$\begin{aligned} \mathscr{B}(V \odot M, W) &\cong \mathscr{B}(V, \mathrm{sHom}(M, W)) \\ \mathscr{B}(M \odot T, U) &\cong \mathscr{B}(T, \mathrm{tHom}(M, U)) \end{aligned}$$

The existence of left and right hom functors defines a *closed bicategory.* Formal definitions and a complete description of closed structures can be found in [MS06].

Notation 2.1. To clarify understanding, we will occasionally make use of the following more explicit notation, familiar from algebraic contexts: For $M : A \nrightarrow B$, $W : A \nrightarrow C$, and $U : D \nrightarrow A$,

$${}_C[\mathrm{Hom}_A({}_B M_A, {}_C W_A)]_B = \mathrm{sHom}(M, W)$$

$${}_A[\mathrm{Hom}_B({}_B M_A, {}_B U_D)]_D = \mathrm{tHom}(M, U)$$

Neither form of the notation is ideal, but it is our hope that using them together will aid readability more than either could alone.

Note that our examples above have internal hom functors, and this defines a closed structure. In general, if $\mathscr{C}_R$ is a bicomplete closed symmetric monoidal category, then the bicategory $\mathscr{B}_R$ built from $\mathscr{C}_R$ is a closed symmetric monoidal bicategory.

2.4. **The bicategorical Yoneda lemma and Morita II.** We remind the reader of the close ties between Morita theory and the bicategorical Yoneda lemma. More detailed discussion is available in [Joh08].

Lemma 2.2 (Yoneda [Str80])**.** *For a pseudofunctor of bicategories $\mathcal{P} : \mathscr{A} \to Cat$, evaluation at the unit 1-cell for each 0-cell, A, of $\mathscr{A}$ provides the components for an equivalence of categories*

$$\Psi_s[\mathscr{A}, Cat](\mathscr{A}(A,-), \mathcal{P}) \xrightarrow{\simeq} \mathcal{P}A.$$

Corollary 2.3 (Morita II)**.**

$$\Psi_s[\mathscr{A}, Cat](\mathscr{A}(A,-), \mathscr{A}(B,-)) \xrightarrow{\simeq} \mathscr{A}(B,A)$$

That is, strong transformations $\mathscr{A}(A,-) \to \mathscr{A}(B,-)$ are given (precisely) by $\odot$-composition with a 1-cell $B \nrightarrow A$. In particular, strong transformations which induce equivalences $\mathscr{A}(A,C) \simeq \mathscr{A}(B,C)$ for all 0-cells C are given by invertible 1-cells $B \nrightarrow A$.

Example 2.4. Let R be a commutative ring; let A and B be R-algebras. Suppose given, for each R-algebra C, a functor $F_C : \mathscr{D}_R(A,C) \to \mathscr{D}_R(B,C)$. Suppose furthermore that, for each D-C-bimodule, K, there is a natural isomorphism $\phi_K : K \otimes_C F_C(X) \cong F_D(K \otimes_C X)$ (in $\mathscr{D}_R$, $\otimes$ denotes the derived tensor). We take these natural isomorphisms to be associative and unital, so that the composite $\phi_L \circ (L \otimes_D \phi_K)$ is equal to $\phi_{L \otimes_D K}$ and ϕ_C factors through the unit isomorphisms $C \otimes_C F(X) \cong F(X) \cong F(C \otimes_C X)$. Then there is an A-B-bimodule M such that $F(X) = X \otimes_A M$.

Of course, this description of strong transformations in $\mathscr{M}_R$ or $\mathscr{D}_R$ makes it clear that Corollary 2.3 may be the beginning of Morita theory in general bicategories, but it is not the end. With this description of strong transformations, one might (perversely) view the classical theorem of Morita as a result which shows that every equivalence of categories $\mathscr{M}_R(A,R) \simeq \mathscr{M}_R(B,R)$ induces a family of functors as above, with all of the necessary naturality and compatibility. Although this is probably not the best way to introduce such a result, it can help explain why obvious generalizations of this result do not hold in DG or topological bicategories. Asking if a naive Morita theorem holds in general is something like asking the following: Supose given two functors, F and G, and, for a specific object R, a morphism $f : F(R) \to G(R)$; does f extend to a natural transformation of functors? When phrased this way, we expect the answer to be negative; the theorems of Morita and Rickard [Ric89] indicate something special about DG algebras concentrated in degree zero.

3. Duality and invertibility in bicategories

To develop general Morita theory more throughly, we first introduce the notions of duality and invertibility. In Subsection 3.2 we give a characterization of Azumaya objects genralizing the classical characterization. For general discussion about duality, we consider fixed 1-cells $X : B \nrightarrow A$ and $Y : A \nrightarrow B$ in a closed bicategory $\mathscr{B}$.

Definition 3.1 (Dual pair)**.** We say (X, Y) is a dual pair, or 'X is left-dual to Y' ('Y is right-dual to X'), or 'X is right-dualizable' ('Y is left-dualizable') to mean that we have 2-cells

$$\eta : A \to X \odot Y \quad \text{and} \quad \varepsilon : Y \odot X \to B$$

such that the following composites are the respective identity 2-cells.

$$X \cong A \odot X \xrightarrow{\eta \odot \mathrm{id}} X \odot Y \odot X \xrightarrow{\mathrm{id} \odot \varepsilon} X \odot B \cong X$$

$$Y \cong Y \odot A \xrightarrow{\mathrm{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \mathrm{id}} B \odot Y \cong Y$$

Definition 3.2 (Base and cobase for a dual pair)**.** When (X, Y) is a dual pair in a bicategory $\mathscr{B}$, we term the source of X (the target of Y) the *base* of the dual pair, and we term the source of Y (the target of X) the *cobase* of the dual pair. Thus, the evaluation map of the dual pair is a two-cell from $Y \odot X$ to the base 1-cell, and the coevaluation (unit) is a two-cell from the cobase 1-cell to $X \odot Y$.

Definition 3.3 (Invertible pair)**.** A dual pair (X, Y) is called invertible if the maps η and ε are isomorphisms. Equivalently, the adjoint pairs described above are adjoint equivalences.

Duality for monoidal categories has been studied at length, and duality in a bicategorical context has been introduced in [MS06, §16.4]. The definition of duality does not require $\mathscr{B}$ to be closed, but we will make use of the following basic facts about duality, some of which do require a closed structure on $\mathscr{B}$.

Proposition 3.4. *A 1-cell $X \in \mathscr{B}(A, B)$ is right-dualizable if and only if the coevaluation*

$$\nu : X \odot (\mathrm{sHom}(X, A)) \to \mathrm{sHom}(X, X)$$

is an isomorphism. Moreover, this is the case if and only if the map

$$\nu_Z : X \odot (\mathrm{sHom}(X, Z)) \to \mathrm{sHom}(X, X \odot Z)$$

is an isomorphism for all 1-cells Z with target A.

Proposition 3.5. *Let (X, Y) be a dual pair in $\mathscr{B}$, with $X : B \nrightarrow A$ and $Y : A \nrightarrow B$.*

i. *For any 0-cell C, we have two adjoint pairs of functors, with left adjoints written on top:*

$$\begin{tikzcd}\mathscr{B}(A, C) \arrow[r, shift left, "- \odot X"] & \mathscr{B}(B, C) \arrow[l, shift left, "- \odot Y"]\end{tikzcd}$$

$$\begin{tikzcd}\mathscr{B}(C, A) \arrow[r, shift left, "Y \odot -"] & \mathscr{B}(C, B) \arrow[l, shift left, "X \odot -"]\end{tikzcd}$$

The structure maps for the dual pair give the triangle identities necessary to show that the displayed functors are adjoint pairs.

ii. *If $\mathscr{B}$ is closed, then Y is canonically isomorphic to* $\mathrm{sHom}(X, B)$*, and for any 1-cell $Z : B \nrightarrow D$, the natural map $Z \odot \mathrm{sHom}(X, B) \to \mathrm{sHom}(X, Z)$ is an isomorphism.*
iii. *Moreover, in the case that $\mathscr{B}$ is closed, X is canonically isomorphic to* $\mathrm{tHom}(Y, A)$*, and for any 1-cell $W : D' \nrightarrow A$, the natural map $\mathrm{tHom}(Y, A) \odot W \to \mathrm{tHom}(Y, W)$ is an isomorphism.*

Lemma 3.6. *Let $X : A \nrightarrow B$ be a 1-cell in $\mathscr{B}(A, B)$. If X is right-dualizable and the unit $B \to X \rhd X$ is an isomorphism, then the evaluation $X \odot (\mathrm{tHom}(X, B)) \to B$ is an isomorphism. Likewise, if X is left-dualizable and the unit $A \to \mathrm{tHom}(X, X)$ is an isomorphism, then the evaluation $(\mathrm{sHom}(X, A)) \odot X \to A$ is an isomorphism.*

Proof. We prove the first statement, leaving the second as an exercise in opposites. Let Y denote the canonical right dual of X. Since X is right-dualizable, Y is left-dualizable and X is isomorphic to the canonical left dual of Y: $X \cong \mathrm{tHom}(Y, A)$. The isomorphism $B \xrightarrow{\cong} \mathrm{sHom}(X, X)$ implies that the unit for the duality is an isomorphism: $B \xrightarrow{\cong} X \odot Y$. Now we have the following commutative square:

$$\begin{tikzcd}X \odot (\mathrm{tHom}(B, X)) \arrow[rr, "\text{evaluation}"] \arrow[d, "\cong"'] & & B \arrow[d, "\cong"] \\ (\mathrm{tHom}(Y, A)) \odot (\mathrm{tHom}(B, X)) \arrow[r, "\cong"] & \mathrm{tHom}(Y, (\mathrm{tHom}(X, B))) \arrow[r, "\cong"] & \mathrm{tHom}((X \odot Y), B)\end{tikzcd}$$

where the two vertical isomorphisms are described above, the left-hand isomorphism is a consequence of dualizability for Y, and the right-hand isomorphism is an exercise in adjunction. □

3.1. **Examples.** The right-dualizable 1-cells in the bicategory $\mathscr{M}_R$ are the finitely-generated projective bimodules. More precisely, they are finitely-generated projective as right-modules over their source (the base of the duality). In $\mathscr{D}_R$, the dualizable objects are the retracts of finite-cell bimodules.

Definition 3.7 (Picard Group)**.** Let A be a 0-cell of a bicategory $\mathscr{B}$. The *Picard group* of A, denoted $Pic(A)$, is the group of isomorphism classes of invertible 1-cells $A \nrightarrow A$.

3.2. **Azumaya objects and the Brauer group.** For the following, we let A be a fixed 0-cell of $\mathscr{D}$. We denote the enveloping 0-cell, $A \otimes_R A^{op}$ by A^e. Let $A_r \in \mathscr{D}(A^e, R)$ denote A regarded as a 1-cell $A^e \nrightarrow R$.

Proposition 3.8. *For A_r as above, the following are equivalent.*

- *i.* $(A_r, \mathrm{sHom}(A_r, A^e))$ *is an invertible pair of 1-cells.*
- *ii.* $(\mathrm{tHom}(A_r, R), A_r)$ *is an invertible pair of 1-cells.*
- *iii.* *a) The evaluation* $(\mathrm{sHom}(A_r, A^e)) \otimes_R A_r \to A^e$ *is an isomorphism.*
  - *b) The coevaluation* $A_r \otimes_{A^e} (\mathrm{sHom}(A_r, A^e)) \to \mathrm{sHom}(A_r, A_r) = {}_R[\mathrm{Hom}_{A^e}({}_RA_{rA^e}, {}_RA_{rA^e})]_R$ *is an isomorphism.*
  - *c) The unit map* $R \to \mathrm{sHom}(A_r, A_r)$ *is an isomorphism.*
- *iv.* *a) The evaluation* $A_r \otimes_{A^e} \mathrm{tHom}(A_r, R) \to R$ *is an isomorphism.*
  - *b) The coevaluation* $(\mathrm{tHom}(A_r, R)) \otimes_R A_r \to \mathrm{tHom}(A_r, A_r) = {}_{A^e}[\mathrm{Hom}_R({}_RA_{rA^e}, {}_RA_{rA^e})]_{A^e}$ *is an isomorphism.*
  - *c) The unit map* $A^e \to \mathrm{tHom}(A_r, A_r)$ *is an isomorphism.*

Proof. In general, a pair of 1-cells (X, Y) is invertible if and only if X is right-dualizable, Y is isomorphic to the canonical right dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *iii*. Likewise, (Y, X) is invertible if and only if X is left-dualizable, Y is isomorphic to the canonical left dual of X, and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *ii* and *iv*. Since (X, Y) is an invertible pair if and only if (Y, X) is so, this finishes the proof. □

Definition 3.9. If A satisfies the four equivalent conditions above, A is an *Azumaya object* of $\mathscr{D}$.

Remark 3.10. Let R be a commutative ring. If $\mathscr{D}$ is the bicategory of R-algebras and their bimodules, this recovers the classical definition of Azumaya algebras over R. We have the following translations in this case:

- *i.* A is *separable* over R if and only if *iii.b* holds.
- *ii.* The center of A is equal to R if and only if *iii.c* holds.
- *iii.* A is *faithfully projective* over R if and only if *iv.a* and *iv.b* hold.

Note, moreover, that in this case *iii.b* and *iii.c* together imply *iii.a*. The proof makes use of ideal theory, and hence we do not expect such a result to hold in general. But we do not have a counterexample.

Question 3.11. Is there a symmetric monoidal bicategory with 0-cell, A, for which *iii.b* and *iii.c* hold, but *iii.a* does not? Note that, in such a case, A_r would be right-dualizable, but not left-dualizable; In algebraic settings, this would mean that A_r is finitely-generated and projective over A^e, but not so over R.

Proposition 3.12. *Let A be a 0-cell of a bicategory $\mathscr{D}$. Then A is Azumaya in $\mathscr{D}$ if and only if there is a 0-cell, B, such that B_r is left-dualizable and there is an invertible 1-cell $P : R \nrightarrow A \otimes_R B$.*

Proof. If A is Azumaya, then taking $B = A^{(op)}$ gives one direction. Given B and P as above, let $P^* = \mathrm{sHom}(P, R)$, and let $P^e = P \otimes_R (P^*)^{op}$, $P^{*e} = P^* \otimes P^{op}$. Then (P^e, P^{*e}) is an invertible pair of 1-cells between the 0-cells R^e and $(A \otimes_R B)^e \cong A^e \otimes_R B^e$. Moreover, since (P, P^*) is an invertible pair, $P^e = P \otimes_R (P^*)^{op} \cong (A \otimes_R B)_r \cong A_r \otimes_R B_r$. Now B_r is a 1-cell $B_r : B^e \nrightarrow R$, and so taking the tensor product with A^e gives $A^e \otimes B_r : A^e \otimes B^e \nrightarrow A^e$ and the unit isomorphism $A_r \odot A^e \cong A_r$ induces $P^e \cong A_r \otimes_R B_r \cong A_r \odot (A^e \otimes_R B_r)$. Thus we have the following diagram of adjunctions, where the left adjoints commute up to natural isomorphism. By uniqueness of adjoints, the right-adjoints therefore also commute up to natural isomorphism.

$$\begin{tikzcd}
\mathscr{D}(R,-) \arrow[r, shift left, "-\odot A_r"] \arrow[dr, shift left, "-\odot P^e"] & \mathscr{D}(A^e,-) \arrow[l, shift left, "\mathrm{sHom}(A_r,-)"] \arrow[d, shift left, "-\odot(A^e\otimes_R B_r)"] \\
& \mathscr{D}(A^e\otimes_R B^e,-) \arrow[ul, shift left, "-\odot P^{*e}"] \arrow[u, shift left, "\mathrm{sHom}((A^e\otimes_R B_r),-)"]
\end{tikzcd}$$

Now the diagonal adjunction is an equivalence, and Lemma 3.6 shows that the evaluation map for $(B_r, \mathrm{sHom}(B_r, B^e))$ is an isomorphism, hence the counit of the vertical adjunction is an isomorphism. The counit for the horizontal adjunction is therefore also an isomorphism and now a diagram chase shows that the units are also isomorphisms. Explicitly, we have:

$$\begin{aligned}
\mathrm{sHom}(A_r,(-\odot A_r)) &\cong \mathrm{sHom}(A_r,(-\odot A_r))\odot P^e\odot P^{*e}\\
&\cong \mathrm{sHom}(A_r,(-\odot A_r))\odot A_r\odot(A^e\otimes_R B_r)\odot P^{*e}\\
&\cong -\odot A_r\odot(A^e\otimes_R B_r)\odot P^{*e}\\
&\cong -\odot P^e\odot P^{*e}
\end{aligned}$$

Since (P^e, P^{*e}) is an invertible pair, this shows that the horizontal adjunction is an equivalence, and therefore A is an Azumaya object. □

Note in the proof above that B is also an Azumaya object; interchanging the roles of A and B we thus have the following.

Corollary 3.13. *For A as above, A is Azumaya if and only if A_r is left-dualizable and there is a 0-cell B with an invertible 1-cell $P: R \nrightarrow B\otimes_R A$.*

Definition 3.14. Let $\mathscr{B}$ be a symmetric monoidal bicategory with unit 0-cell R. The *Brauer group* of R, denoted $Br(R)$, is the group of 1-cell-equivalence-classes of 0-cells A for which there exists a 0-cell B such that $A\otimes_R B$ is equivalent to R.

Remark 3.15. The Azumaya objects are the objects $A \in Br(R)$ such that A_r is left-dualizable. Classically, 1-cell-equivalence is Morita equivalence, and if R is a commutative ring, every $A \in Br(R)$ satisfies the left-dualizability condition. (In this context, it means that A is finitely-generated and projective as an R-module.) This is the same issue noted in Remark 3.10.

Question 3.16. Is there a symmetric monoidal bicategory with unit R and two 0-cells, A, B, for which $A_r\otimes_R B_r$ is invertible but neither A_r nor B_r is invertible?

4. TRIANGULATED BICATEGORIES

We recall first the definitions of localizing subcategory and generator for a triangulated category, and then give a definition (4.4) of triangulated bicategory suitable for our purposes. In particular, under this definition $\mathscr{D}_R$ are triangulated bicategories.

Definition 4.1 (Localizing subcategory).
If $\mathscr{T}$ is a triangulated category with infinite coproducts, a *localizing* subcategory, $\mathscr{S}$, is a full triangulated subcategory of $\mathscr{T}$ which is closed under coproducts from $\mathscr{T}$.

Remark 4.2. This is equivalent to the definition for arbitrary triangulated categories of [Hov99], (which requires that a localizing subcategory be thick) because a triangulated subcategory automatically satisfies the 2-out-of-3 property and because in any triangulated category with countable coproducts, idempotents have splittings. See [Nee01, 1.5.2, 1.6.8, and 3.2.7] for details.

Definition 4.3 (Triangulated generator)**.**
A set, $\mathscr{P}$, of objects in $\mathscr{T}$ (triangulated category with infinite coproducts, as above) is a set of *triangulated generators* (or simply *generators*) if the only localizing subcategory containing $\mathscr{P}$ is $\mathscr{T}$ itself.

Definition 4.4 (Triangulated bicategory [MS06, §16.7])**.**
A closed bicategory $\mathscr{B}$ will be called a *triangulated bicategory* if for each pair of 0-cells, A and B, $\mathscr{B}(A,B)$ is a triangulated category with infinite coproducts, and if the suspension, Σ, is a pseudofunctor on $\mathscr{B}$, and furthermore the local triangulations on $\mathscr{B}$ are compatible as described in the following two axioms.

(TC1) For a 1-cell $X : A \nrightarrow B$, there is a natural isomorphism

$$\alpha : X \odot \Sigma A \to \Sigma X$$

such that the composite below is multiplication by -1.

$$\Sigma^2 A = \Sigma(\Sigma A) \xrightarrow{\alpha^{-1}} \Sigma A \odot \Sigma A \xrightarrow{\gamma} \Sigma A \odot \Sigma A \xrightarrow{\alpha} \Sigma(\Sigma A) = \Sigma^2 A$$

(TC2) For any 1-cell, W, the functors $W \odot -$, $- \odot W$, $W \rhd -$, and $- \rhd W$ are exact.

If $\mathscr{B}$ is a triangulated bicategory and P, Q are 1-cells in $\mathscr{B}(A,B)$, we emphasize that $\mathscr{B}$ is triangulated by writing the abelian group of 2-cells $P \to Q$ as $\mathscr{B}[P,Q]$ and by writing the graded abelian group obtained by taking shifts of Q as $\mathscr{B}[P,Q]_*$. To emphasize the source and target of P and Q, we may also write $\mathscr{B}(A,B)[P,Q]_*$.

Definition 4.5 ($\odot$-faithful 1-cells)**.**
In any locally additive bicategory, $\mathscr{B}$, a 1-cell $W : A \nrightarrow B$ is called *left-faithful* if triviality for any 1-cell $Z : C \nrightarrow A$ is detected by triviality of the composite $W \odot Z$. That is, $Z : C \nrightarrow A$ is zero if and only if $W \odot Z = 0$. A collection of 1-cells, $\mathscr{E}$, in $\mathscr{B}(A,B)$ is called *jointly left-faithful* if the objects have this property jointly; that is, $Z = 0$ if and only if $W \odot Z = 0$ for all $W \in \mathscr{E}$. The term *left-faithful* is defined similarly, considering $- \odot W$ instead of $W \odot -$.

Remark 4.6. If $\mathscr{B}$ is a monoidal additive category with monoidal product $\odot$, the unit object is both left- and right-faithful. In arbitrary locally additive bicategories, if $A \neq B$ then $\mathscr{B}(A,B)$ may not have a single object with this property, but in relevant examples the collection of all 1-cells, $\mathrm{ob}\mathscr{B}(A,B)$, does have this property jointly. As a counter-point to this remark, we have the following lemma.

Lemma 4.7. *Let $\mathscr{B}$ be a triangulated bicategory, and let $P : A \nrightarrow B$ be a generator for $\mathscr{B}(A,B)$. If the collection of all 1-cells, $\mathscr{B}(A,B)$, is jointly left-faithful (resp. right-faithful), then P is left-faithful (resp. right-faithful).*

Proof. Consider the left-faithful case; the right-faithful case is similar. Given any 1-cell $Z : C \nrightarrow A$ with $P \odot Z = 0$, let $\mathscr{S}$ be the full subcategory of 1-cells, $W : A \nrightarrow B$ for which $W \odot Z = 0$. This is a localizing subcategory of $\mathscr{B}(A,B)$, and by assumption $P \in \mathscr{S}$, so $\mathscr{S} = \mathscr{B}(A,B)$, and hence $Z = 0$. □

Remark 4.8. Since the functors $P \odot -$ are exact, the property of $P \odot -$ detecting trivial objects is equivalent to $P \odot -$ detecting isomorphisms (meaning that a 2-cell f is an isomorphism if and only if $P \odot f$ is so).

4.1. **Tilting theory.** For this subsection, we let $\mathscr{D}$ denote a closed, symmetric monoidal bicategory with unit 0-cell R. We do not require that $\mathscr{D}$ be triangulated, but simply that $\mathscr{D}$ have a 0-object in each 1-cell category.

Definition 4.9. Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}(A,B)$.

A 1-cell $M : C \nrightarrow A$ is *left-T-acyclic* if $T \odot M = 0$. A 1-cell $N : C \nrightarrow A$ is *left-T-local* if $\mathscr{D}(C,A)[M,N]_* = 0$ for all T-acyclic 1-cells $M \in \mathscr{D}(C,A)$. The full subcategory of left-T-local 1-cells in $\mathscr{D}(C,A)$ is denoted $\mathscr{D}(C,A)_{\langle T_\odot \rangle}$. (The notation $T_\odot$ is intended to remind the reader of push-forward via $\odot$-composition.)

A 1-cell $M' : B \nrightarrow C$ is *right-T-acyclic* if $M \odot T = 0$. A 1-cell $N' : B \nrightarrow C$ is *right-T-local* if $\mathscr{D}(B,C)[M',N']_* = 0$ for all right-T-acyclic 1-cells $M' \in \mathscr{D}(B,C)$. The full subcategory of right-T-local 1-cells in $\mathscr{D}(B,C)$ is denoted $\mathscr{D}(B,C)_{\langle T^\odot \rangle}$.(The notation $T^\odot$ is intended to remind the reader of pull-back via $\odot$-composition.)

Baker and Lazarev describe the following in the context of spectra, but their methods generalize to our setting. The key observation is that for any 1-cell P whose source is A, $\mathrm{sHom}(T,P)$ is right-T-local. Likewise, if P' is any 1-cell whose target is B, $\mathrm{tHom}(T,P')$ is left-T-local.

Proposition 4.10 (Baker-Lazarev factorization [BL04]). *Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}(A,B)$. The adjunctions induced by T factor through the T-local pseudofunctors; we have the following diagrams of adjoint transformations:*

$$\begin{tikzcd}
\mathscr{D}(B,-) \arrow[rr, shift left, harpoon, "-\odot T"] \arrow[dr, shift left, harpoon] & & \mathscr{D}(A,-) \arrow[ll, shift left, harpoon, "\mathrm{sHom}(T,-)"] \arrow[dl, shift left, harpoon, "\mathrm{sHom}(T,-)"] \\
& \mathscr{D}(B,-)_{\langle T^{\odot} \rangle} \arrow[ul, shift left, harpoon] \arrow[ur, shift left, harpoon, "-\odot T"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{D}(-,A) \arrow[rr, shift left, harpoon, "T\odot -"] \arrow[dr, shift left, harpoon] & & \mathscr{D}(-,B) \arrow[ll, shift left, harpoon, "\mathrm{tHom}(T,-)"] \arrow[dl, shift left, harpoon, "\mathrm{tHom}(T,-)"] \\
& \mathscr{D}(-,A)_{\langle T_{\odot} \rangle} \arrow[ul, shift left, harpoon] \arrow[ur, shift left, harpoon, "T\odot -"] &
\end{tikzcd}$$

Proposition 4.11 ([BL04]). *If a 1-cell $T \in \mathscr{D}(A,E)$ is right-dualizable and the unit map induces an isomorphism $E \cong \mathrm{sHom}(T,T) = {}_E[\mathrm{Hom}_A({}_ET_A, {}_ET_A)]_E$, then the induced adjoint pair is an equivalence $\mathscr{D}(-,A)_{\langle T_{\odot} \rangle} \simeq \mathscr{D}(-,E)$. We have a corresponding statement for the case of left-dualizability.*

Proof. Let T^* denote the right-dual to T. Since T is right-dualizable, T^* is left-dualizable and the evaluation map $T \odot (\mathrm{tHom}(T,E)) \to E$ is an isomorphism (Lemma 3.6). Moreover, $\mathrm{tHom}(T,-)$ takes values in the T-local category and hence the fact that the unit of the adjunction is an isomorphism follows from the fact that the evaluation is so. ☐

Corollary 4.12. *If T satisfies the hypotheses of 4.11 and if in addition T is left-faithful (Definition 4.5), then all three of the adjoint pairs above are equivalences.*

Proof. If T is left-faithful, then all left-T-acyclics are trivial, and hence $\mathscr{D}(-,A) \simeq \mathscr{D}(-,A)_{\langle T_{\odot} \rangle}$. The result then follows from 4.11. ☐

Proposition 4.13. *Let $T : A \nrightarrow B$ be a 1-cell in $\mathscr{D}$. The following are equivalent:*

i. *T is invertible.*
ii. *a) T is right-dualizable.*
  b) The unit induces $B \cong \mathrm{sHom}(T,T)$.
  c) A is left-T-local.
iii. *a) T is left-dualizable.*
  b) The unit induces $A \cong \mathrm{tHom}(T,T)$.
  c) B is right-T-local.

Let A be a 0-cell of $\mathscr{D}$ and (as above) let A_r denote A regarded as a 1-cell $A^e \nrightarrow R$. Applying the above proposition to A_r, we have the following. For comparison with [BL04], note that in topological contexts $\mathrm{sHom}(A_r, A_r)$ is denoted $THH_R(A,A)$.

Corollary 4.14 ([BL04, 2.1,2.3]). *The following are equivalent:*

i. *A is Azumaya in $\mathscr{D}$.*
ii. *a) A_r is right-dualizable.*
  b) The unit induces $R \cong \mathrm{sHom}(A_r, A_r)$.
  c) A^e is left-A_r-local.
iii. *a) A_r is left-dualizable.*
  b) The unit induces $A^e \cong \mathrm{tHom}(A_r, A_r)$.
  c) R is right-A_r-local.

Remark 4.15. In [BL04], Baker and Lazarev prove the equivalence of the second two conditions above–clearly understanding them as a good definition of Azumaya spectra. The formal connection to classical definitions, however, is new.

A bicategorical perspective on the following two results is developed in [Joh08]. Our work with invertibility here allows us to give more streamlined proofs.

Corollary 4.16 (Rickard). *Let S be a DG R-algebra, and let T be a DG S-module. If T has the following two properties, then $\mathscr{D}_R(S)$ and $\mathscr{D}_R(End_S(T))$ are equivalent as triangulated categories.*

i. *T is a right-dualizable S-module.*
ii. *T generates the triangulated category $\mathscr{D}_R(S)$.*

Proof. Let $\widetilde{T}$ denote T regarded as a bimodule over $E = {}_R[\mathrm{Hom}_S({}_RT_S, {}_RT_S)]_R$. Since T is (right-)dualizable, $\widetilde{T}$ is (right-)dualizable in $\mathscr{D}_R(S, E)$.

Since R is the unit of the symmetric monoidal bicategory $\mathscr{D}_R$, the 1-cells of $\mathscr{D}_R(S, R)$ are jointly left-faithful (Definition 4.5). Hence Lemma 4.7 shows that T is left-faithful. This means that $\widetilde{T}$ is also left-faithful, and thus the evaluation $(\widetilde{T} \triangleright S) \odot \widetilde{T} \to S$ is an isomorphism in $\mathscr{D}_R(S, S)$: The composite below is the identity and the first map, induced by the unit of the adjunction, is an isomorphism so the second must be also.

$$\widetilde{T} \xrightarrow{\cong} \widetilde{T} \odot (\widetilde{T} \triangleright S) \odot \widetilde{T} \xrightarrow{1 \odot \mathrm{eval}} \widetilde{T}$$

□

Note. The object T above (or below) is called a *tilting complex* (or *spectrum*).

Corollary 4.17. *Let R be a commutative ring spectrum, and let $\mathscr{D}_R$ denote the bicategory of R-algebras and homotopy categories of bimodules. Suppose A is an R-algebra, and let T be a fibrant and cofibrant A-module, with endomorphism R-algebra $E = F_A(T, T)$. If T has the following two properties, then $\mathscr{D}_R(A)$ and $\mathscr{D}_R(E)$ are equivalent categories.*

i. T is (right-)dualizable as an A-module.
ii. T generates the triangulated category $\mathscr{D}_R(A)$.

Notation 4.18. Given a map of R-algebras $\iota : B \to E$, we have two restriction-of-scalars functors: one for restriction of left modules, and another for restriction of right modules. For any R-algebra A, We let $\iota_\ell^* : \mathscr{S}_R(A, E) \to \mathscr{S}_R(A, B)$ denote restriction on the left (target), and $\iota_r^* : \mathscr{S}_R(E, A) \to \mathscr{S}_R(B, A)$ denote restriction on the right (source). Both functors create weak-equivalences and fibrations.

Proof of 4.17. Let $E = \mathrm{sHom}(T, T) = F_A(T, T)$. The unit map $R \to E$ is obtained as the composite of algebra maps $R \to B \to E$. Let $\widetilde{T}$ be a cofibrant replacement for T in $\mathscr{S}_R(A, E)$. Recall that T is cofibrant in $\mathscr{S}_R(A, B)$, and hence has the LLP with respect to acyclic fibrations. We construct $\widetilde{T}$ by the usual factorization of the map from the initial object, and the forgetful functor ι_ℓ^* creates weak equivalences and fibrations, so the lifting property for T gives a weak equivalence $T \xrightarrow{\simeq} \iota_\ell^* \widetilde{T}$.

The canonical dual of T is $F_A(T, A) = \mathrm{sHom}(T, A) \in \mathscr{S}_R(B, A)$, and we let D denote a cofibrant replacement for $F_A(T, A)$ in $\mathscr{S}_R(B, A)$, so that we have a weak equivalence $D \xrightarrow{\simeq} F_A(T, A)$. The canonical dual of T has a right-action of the endomorphism k-algebra, E, and we let $\widetilde{D}$ be a cofibrant replacement for $F_A(T, A)$ in $\mathscr{S}_R(E, A)$, constructed again by the usual factorization. Since the forgetful functor ι_r^* creates weak equivalences and fibrations, we have an acyclic fibration $\iota_r^* \widetilde{D} \xrightarrow{\simeq} F_A(T, A)$ in $\mathscr{S}_R(B, A)$. Because D is cofibrant, the weak equivalence $D \xrightarrow{\simeq} F_A(T, A)$ lifts with respect to acyclic fibrations and hence we have a weak equivalence $D \xrightarrow{\simeq} \iota_r^* \widetilde{D}$.

Now we show that $(\widetilde{T}, \widetilde{D})$ is a dual pair in $\mathscr{D}_R$. The weak equivalences $\widetilde{T} \to T$ and $\widetilde{D} \to F_A(T, A)$ in $\mathscr{S}_R(A, E)$ and $\mathscr{S}_R(E, A)$, respectively, give maps

$$\widetilde{T} \odot \widetilde{D} \to T \odot F_A(T, A) \to E \text{ and } \widetilde{D} \odot \widetilde{T} \to F_A(T, A) \odot T \to A$$

in $\mathscr{S}_R(E, E)$ and $\mathscr{S}_R(A, A)$, respectively. Moreover, the first map is an isomorphism in $\mathscr{D}_R(E, E)$ because its image under $\iota_\ell^* \iota_r^*$ is a composite of two isomorphisms in $\mathscr{D}_R(B, B)$:

$$\iota_\ell^* \widetilde{T} \odot \iota_r^* \widetilde{D} \cong T \odot D \cong \iota_\ell^* \iota_r^* E.$$

The inverse to this map gives the unit for the dual pair, and the duality diagrams commute because the corresponding diagrams for T and $F_A(T, A)$ do. Hence the functors $- \odot \widetilde{T}$ and $- \odot \widetilde{D}$ induce an adjunction

$$\begin{tikzcd} \mathscr{D}_R(A, C) \arrow[r, shift left, "- \odot \widetilde{T}"] & \mathscr{D}_R(E, C) \arrow[l, shift left, "- \odot \widetilde{D}"] \end{tikzcd}$$

and the unit of this adjunction is an isomorphism.

As in the algebraic case, the 1-cells of $\mathscr{D}_R(A, R)$ are jointly left-faithful and hence the generator T is left-faithful. Since ι_ℓ^* creates weak equivalences, $\widetilde{T}$ is also left-faithful and the result follows just as in the algebraic case. □

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