Obstruction theory for E_∞ maps

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Two main points

- Obstruction theory for T-algebra maps,
  where T is a monad on a topological category $\mathscr{C}$.
- Demonstrations in rational homotopy
  when T is an E_∞ monad on $\mathscr{S}pectra$.

A toy question

Suppose A and B are commutative d.g. algebras over $\mathbb{Q}$:

$$\cdots \leftarrow A_2 \leftarrow A_1 \leftarrow A_0 \leftarrow A_{-1} \leftarrow \cdots$$

- $|xy| = |x| + |y|$
- $d(xy) = (dx)y + (-1)^{|x|}x(dy)$
- $xy = (-1)^{|x|\,|y|}yx$.

Then H^*A and H^*B are (graded-)commutative $\mathbb{Q}$-algebras.

A toy question

Let $A \xrightarrow{f} B$ be a map of chain complexes such that

$$H^*A \xrightarrow{f^*} H^*B$$

is a map of (graded-)commutative $\mathbb{Q}$-algebras.

Question

Is f a commutative d.g. algebra map? Is it chain homotopic to one?

Answer

Not always

A toy question

Let $A \xrightarrow{f} B$ be a map of chain complexes such that

$$H^*A \xrightarrow{f^*} H^*B$$

is a map of (graded-)commutative $\mathbb{Q}$-algebras.

Question

Is f a commutative d.g. algebra map? Is it chain homotopic to one?

Better Answer

Develop an obstruction theory to analyze f. Basic idea: Take

$$\mathbb{P}^{\bullet+1}A \to A$$

a simplicial resolution of A by free commutative d.g. $\mathbb{Q}$-algebras. Consider the cosimplicial set $\mathcal{C}omm\ \mathbb{Q}\text{-}\mathcal{alg}(\mathbb{P}^{\bullet+1}A, B) \ldots$

More serious question(s)

Let A and B be E_∞ ring spectra, and let

$$A \xrightarrow{f} B$$

be a map of underlying spectra.

Question(s)

Is f H_∞?
If so, does it rigidify to an E_∞ map?
If so, is the rigidification unique?

Main demonstration

Let X and Y be spaces. There is an E_∞ mapping spectrum $H\mathbb{Q}^X$:

$$\pi_* H\mathbb{Q}^X = H^*(X;\mathbb{Q}). \quad \text{(a graded } \mathbb{Q}\text{-algebra)}$$

Consider maps of spectra

$$H\mathbb{Q}^X \xrightarrow{f} H\mathbb{Q}^Y$$

such that $\pi_* f$ induces a commutative $\mathbb{Q}$-algebra map

$$H^*(X;\mathbb{Q}) \xrightarrow{\pi_* f} H^*(Y;\mathbb{Q}).$$

Question

Is f homotopic to an E_∞ map?

Note: A map of spaces $Y \to X$ induces an E_∞ map

$$H\mathbb{Q}^X \to H\mathbb{Q}^Y.$$

Main demonstration

Let $Y = S^2$ and let $X = N =$ the Heisenberg nilmanifold:

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \Big/ \text{integer entries}$$

$$S^1 \longrightarrow N \longrightarrow T^2$$

$$* \longrightarrow \mathbb{Z} \longrightarrow \pi_1 N \longrightarrow \mathbb{Z}^2 \longrightarrow *$$

$$\pi_r N = 0 \text{ for } r > 1$$

$$H^*(N;\mathbb{Q}) = \Lambda(x_1, y_1, \alpha_2, \beta_2) \Big/ \begin{matrix} xy=0, \\ \alpha^2=\beta^2=0, \\ x\alpha=y\beta=0, \\ x\beta+y\alpha=0 \end{matrix}$$

$$H^*(S^2;\mathbb{Q}) = \Lambda(e_2)/e^2$$

Consider maps $H\mathbb{Q}^N \to H\mathbb{Q}^{S^2}$ dual to α or β ...

Monads

Let T be a monad on $\mathscr{C}$:

$$T\colon \mathscr{C} \longrightarrow \mathscr{C}$$

$$\text{(unit) } \mathrm{Id} \longrightarrow T$$

$$\text{(mult.) } T^2 \xrightarrow{\mu} T$$

An object X is a T-algebra if there is a structure map $TX \xrightarrow{\sigma} X$ compatible with unit and multiplication structure of T.

$$\begin{array}{ccc} T^2X & \xrightarrow{T\sigma} & TX \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\sigma} \\ TX & \xrightarrow{\sigma} & X \end{array}$$

E.g. X is a set, TX is the free group on X

X is a group if there is a structure map $TX \xrightarrow{\sigma} X$ such that …

Example: Monad arising from an E_∞ operad

Let $\mathscr{O}$ be an E_∞ operad on spectra, and let $T = \mathbb{P}$ be the associated monad:

$\mathcal{h}o(\mathscr{C}_\mathbb{P})$ is the homotopy category of E_∞ ring spectra

Let $\mathcal{h}\mathbb{P}$ be the induced monad on $\mathcal{h}o\mathscr{C}$:

$(\mathcal{h}o\mathscr{C})_{\mathcal{h}\mathbb{P}}$ is the category of H_∞ ring spectra

There are forgetful functors $\mathcal{h}o(\mathscr{C}_\mathbb{P}) \to (\mathcal{h}o\mathscr{C})_{\mathcal{h}\mathbb{P}} \to \mathcal{h}o\mathscr{C}$.

Rephrased Question(s)

Are these functors full?
Are these functors faithful?

Other examples

- Monads encoding group action on spaces or spectra
- Descent monads for commutative ring map $k \to K$
- Monads arising from A_∞ operads, E_n operads
- Your favorite topological monad!

There are forgetful functors $\mathcal{ho}(\mathscr{C}_T) \to (\mathcal{ho}\mathscr{C})_{\mathcal{h}T} \to \mathcal{ho}\mathscr{C}$.

Rephrased Question(s)

Are these functors full?
Are these functors faithful?

The simplicial resolution

Suppose that A and B are T-algebras in $\mathscr{C}$. Successively applying T yields a simplicial object $T^{\bullet+1}A$.

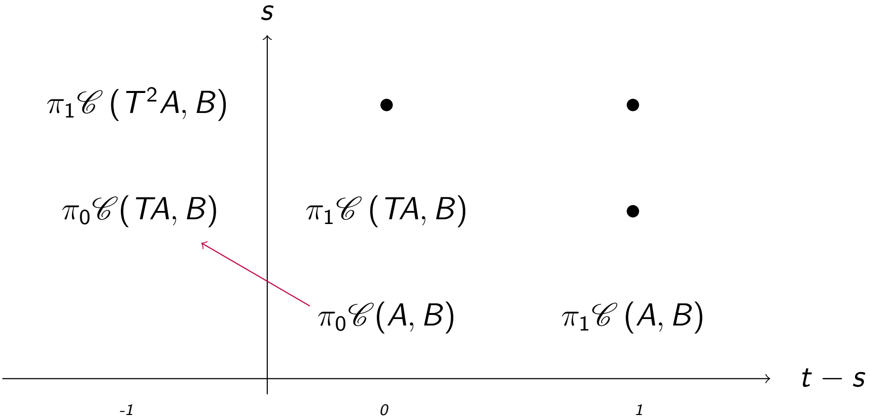
$\mathscr{C}_T(T^{s+1}A, B)$ is the space of T-algebra maps $\mathscr{C}_T(T^{s+1}A, B) \cong \mathscr{C}(T^s A, B)$. May be empty!

$\mathscr{C}_T(T^{\bullet+1}A, B) \cong \mathscr{C}(T^\bullet A, B)$ is a cosimplicial space
$\rightsquigarrow$ tower of fibrations, Bousfield-Kan spectral sequence under General Assumptions.

General Assumptions

- $\mathscr{C}$ is a topologically enriched model category.
- T is a topological monad on $\mathscr{C}$.
- $\mathscr{C}_T$ has an induced topological model category structure such that all limits and tensors are calculated in $\mathscr{C}$ as well as all sifted colimits.
- $A \in \mathscr{C}_T$ is such that $T^{\bullet+1}A$ is Reedy cofibrant in $s\mathscr{C}_T$.
- One of the following:
  - $T^{\bullet+1}A$ is Reedy cofibrant in $s\mathscr{C}$.
  - T commutes with geometric realization of simplicial T-algebras.

An observation on $E_1 = E_1(f)$



Observation: d_1 is the difference around $\begin{array}{ccc} TA & \to & TB \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$ in $ho\mathscr{C}$.

Theorem

Let T be a monad on $\mathscr{C}$ satisfying the General Assumptions and let $\hbar T$ be the induced monad on $\hbar o\mathscr{C}$.

For B in $\mathscr{C}_T$ there is a fringed Bousfield-Kan spectral sequence $E_r^{s,t}$:

1. $E_1^{0,0} = \pi_0\mathscr{C}(A,B) = \hbar o\mathscr{C}(A,B)$.
2. A homotopy class $[f] \in E_1^{0,0}$ survives to $E_2^{0,0}$ if and only if $[f]$ is an $\hbar T$-algebra map, that is,
$$E_2^{0,0} = (\hbar o\mathscr{C})_{\hbar T}(A,B).$$
3. When the $E_2 = E_2(f)$ page of the obstruction spectral sequence is defined, we have
$$E_2^{s,t} = \pi^s\pi_t\left(\mathscr{C}(T^\bullet A, B), f\right).$$

Theorem

Let T be a monad on $\mathscr{C}$ satisfying the General Assumptions and let $\mathcal{h}T$ be the induced monad on $\mathcal{ho}\mathscr{C}$.

For B in $\mathscr{C}_T$ there is a fringed Bousfield-Kan spectral sequence $E_r^{s,t}$:

4. The prospective basepoint f survives to the E_∞ page if and only if f lifts to a T-algebra map. In this case, the spectral sequence conditionally converges to $\pi_*(\mathscr{C}_T(A,B), f)$. $(E_\infty = E_\infty\ !)$

5. The edge maps

$$\pi_0 \mathscr{C}_T(A,B) \longrightarrow E_2^{0,0} = (\mathcal{ho}\mathscr{C})_{\mathcal{h}T}(A,B) \longrightarrow E_1^{0,0} = \pi_0 \mathscr{C}(A,B).$$

are the forgetful functors from T-algebras to $\mathcal{h}T$-algebras to the homotopy category of $\mathscr{C}$, respectively.

Corollaries, $T = \mathbb{P}$

Consider

$$E_\infty(A, B) \xrightarrow{\text{forget}} H_\infty(A, B).$$

Corollary

The forgetful functor from
the homotopy category of E_∞ ring spectra to H_∞ ring spectra
is faithful if and only if $E_\infty^{t,t} = 0$ for $t > 0$.

Corollary

The forgetful functor from
the homotopy category of E_∞ ring spectra to H_∞ ring spectra
is full if and only if the differential d_r on $E_r^{0,0}$ is trivial for all $r \geq 2$.

Demonstrations

For pointed spaces X and Y, recall

$$\pi_* H\mathbb{Q}^X \cong H^*(X;\mathbb{Q})$$

$$H_\infty(H\mathbb{Q}^X, H\mathbb{Q}^Y) \cong \mathcal{C}omm\,\mathbb{Q}\text{-}\mathcal{alg}(H^*(X;\mathbb{Q}), H^*(Y;\mathbb{Q}))$$

Consider the forgetful functor

$$E_\infty(H\mathbb{Q}^X, H\mathbb{Q}^Y) \longrightarrow H_\infty(H\mathbb{Q}^X, H\mathbb{Q}^Y).$$

Note: there is a natural base point for the obstruction spectral sequence

$$\varepsilon\colon H\mathbb{Q}^X \to H\mathbb{Q} \to H\mathbb{Q}^Y$$

induced by $X \to * \to Y$.

The Hopf map $\eta\colon S^3 \to S^2$

The Hopf map induces an E_∞ map

$$H\mathbb{Q}^{S^2} \to H\mathbb{Q}^{S^3}.$$

The rational cohomology of S^2

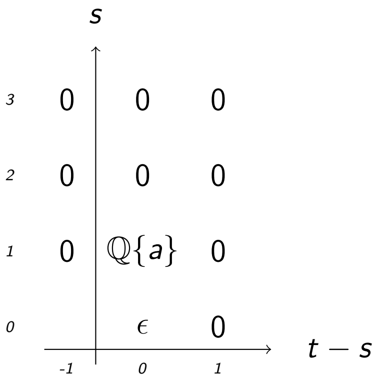
$$H^*(S^2;\mathbb{Q}) = \Lambda(e_2)/e^2$$

has resolution

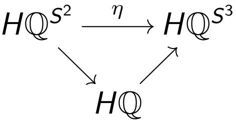
$$R = \Lambda(a_3, e_2),\ da = e^2.$$

The dual map $a = \eta\colon R \to H^*(S^3;\mathbb{Q})$ is a commutative $\mathbb{Q}$-algebra map.

The Hopf map, E_2 page



The Hopf map induces a nontrivial E_∞ map which is in the kernel of the forgetful functor.
The diagram does not commute in $\mathcal{ho}\, E_\infty$,
but does commute in H_∞.



The Heisenberg nilmanifold

Consider $E_\infty(H\mathbb{Q}^N, H\mathbb{Q}^{S^2})$.

$$N = \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \Big/ \text{integer entries}$$

$$S^1 \longrightarrow N \longrightarrow T^2$$

$$* \longrightarrow \mathbb{Z} \longrightarrow \pi_1 N \longrightarrow \mathbb{Z}^2 \longrightarrow *$$

$$H^*(N;\mathbb{Q}) = \Lambda(x_1, y_1, \alpha_2, \beta_2) \Big/ \begin{matrix} xy=0, \\ \alpha^2=\beta^2=0, \\ x\alpha=y\beta=0, \\ x\beta+y\alpha=0 \end{matrix}$$

$$H^*(S^2;\mathbb{Q}) = \Lambda(e_2)/e^2$$

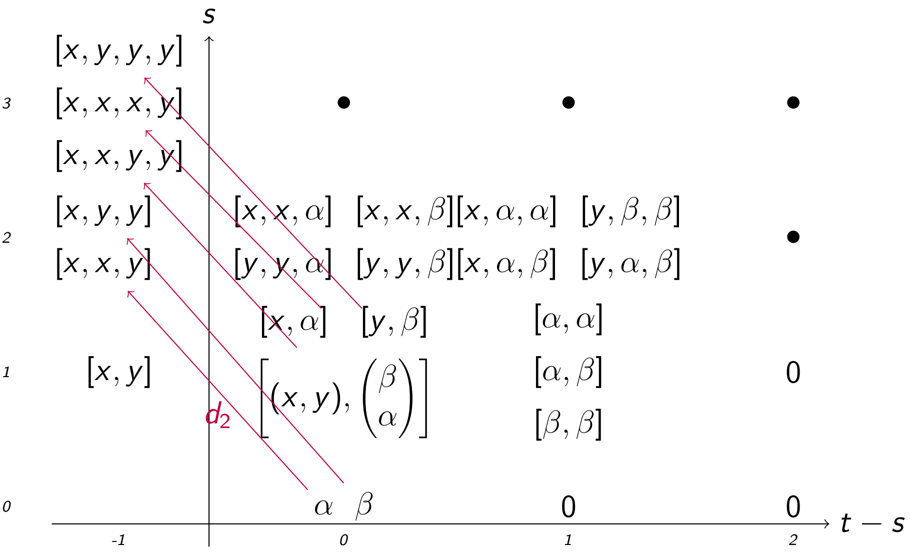
The Heisenberg nilmanifold

The dual maps $\alpha, \beta \colon H^*(N;\mathbb{Q}) \to H^*(S^2;\mathbb{Q})$ survive to $E_2^{0,0}$ and thus are H_∞ maps.

However they cannot be (homotopic to) E_∞ maps because α and β are Massey products $\langle x, y, y\rangle$ and $\langle y, x, x\rangle$; these must vanish under an E_∞ map for degree reasons.

Thus the forgetful functor fails to be full.

The Heisenberg nilmanifold, E_2 page



Conclusion

- Obstruction theory for T-algebra maps, where T is a monad on a topological category $\mathscr{C}$.

    Obstruction spectral sequence to analyze whether maps lift to $\mathcal{h}T$-algebra maps or T-algebra maps

- Demonstrations in rational homotopy when T arises from an E_∞ operad on spectra.

    Examples showing failure of forgetful functor to be full or faithful

Thank You!